The Method of Equivalence

I will now discuss a fundamental method, due to Élie Cartan, for computing the differential invariants of a geometric structure on a manifold.

1. The Method of Equivalence

Shortly after 1900, Élie Cartan developed a uniform method for analysing the differential invariants of many geometric structures, nowadays called the 'method of equivalence'. In this section, I will describe the method and some of the basic results. In the following section, I will illustrate the method by numerous examples.

1.1. **The coframe bundle.** Let M be a smooth n-manifold. A *coframe* at $x \in M$ is a linear isomorphism $u : T_xM \to \mathbb{R}^n$.[1] In practice, I find it helpful to think of a coframe at x as a 1-jet of a coordinate system centered at x. The set of such coframes based at x will be denoted by $F^*_x(M)$ (or simply F^*_x when the manifold M is clear from context). The disjoint union of the F^*_x as x varies on M will be denoted $F^*(M)$ (or, again, simply F^*) and is called the space of general coframes of M. The *basepoint mapping* $\pi : F^* \to M$ is defined by $\pi(F^*_x) = x$. The group $\mathrm{GL}(n,\mathbb{R})$ acts on F^* on the right by the rule $u \cdot A = A^{-1}u$ for $u \in F^*$ and $A \in \mathrm{GL}(n,\mathbb{R})$. This action is simply transitive on each of the π-fibers F^*_x.

For any open set $U \subset M$, a (smooth) *coframing* of U is a choice $\eta = (\eta^i)$ where $1 \le i \le n$ and the η^i are n (smooth) 1-forms on U that are everywhere linearly indepdendent. Associated to such a coframing η, there is a map $H : U \times \mathrm{GL}(n,\mathbb{R}) \to F^*(U)$ defined by the formula

$$H(x, A) = A^{-1}\eta_x\,.$$

This map respects the right action by $\mathrm{GL}(n,\mathbb{R})$, i.e., $H(x, AB) = B^{-1}\,H(x, A) = H(x, A)\cdot B$. There is a unique smooth structure on F^* for which the maps H so constructed are diffeomorphisms. Indeed, there is a unique structure of a smooth, principal $\mathrm{GL}(n,\mathbb{R})$-bundle on F^* so that the inverses of these maps are its smooth trivializations. Henceforth, this will be the smooth structure I assume on F^*. The basepoint mapping $\pi : F^* \to M$ is then a smooth submersion and a smooth local section of F^* is simply a coframing on the domain of the section.

If N is another smooth n-manifold and $f : M \to N$ is a local diffeomorphism, then a smooth bundle map $f_1 : F^*(M) \to F^*(N)$ covering f is defined by the rule

$$f_1(u) = u \circ \bigl(f'\bigl(\pi(u)\bigr)\bigr)^{-1}\,.$$

[1] Here and elsewhere, I always assume that $\mathbb{R}^n$ is represented as *columns* of height n with real entries. This is necessary in order that the representation of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ be via standard matrix multiplication. The neglect of this seemingly trivial point has caused considerable confusion, which I hope to avoid.

The assignment $f \mapsto f_1$ is functorial (and covariant), as expected, with a commutative diagram

$$\begin{array}{ccc} F^*(M) & \xrightarrow{f_1} & F^*(N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

and f_1 is a diffeomorphism.

Finally, I should remark that is it sometimes more useful (and conceptually clearer) to replace $\mathbb{R}^n$ by an abstract n-dimensional real vector space V. One then speaks of the V-valued coframes, which are isomorphisms $u : T_xM \to V$, and constructs the principal right $\mathrm{GL}(V)$-bundle $F^*(M, V)$ of V-valued coframes. The chief advantage of this more general notation is that it is easier to keep track of the fundamental distinction between V and V^*. I will resort to this when necessary.

1.1.1. *G-structures.* Let G be an n-by-n matrix group, i.e., a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A (smooth) G-structure on an n-manifold M is simply a (smooth) G-subbundle of $F^* = F^*(M)$, i.e., a (smooth) submanifold $B \subset F^*$ so that the restricted basepoint mapping $\pi : B \to M$ is a surjective submersion whose fibers $B_x = B \cap F_x$ are G-orbits.

When G is closed in $\mathrm{GL}(n, \mathbb{R})$, an alternative definition is available, for then the quotient space F^*/G carries the structure of a smooth bundle over M. Its fibers are essentially copies of the homogeneous space $\mathrm{GL}(n, \mathbb{R})/G$. A choice of a (smooth) G-structure on M is then equivalent to a choice of a (smooth) section of this bundle. This viewpoint is frequently useful when one wants to make statements about the space of G-structures, as I will. Since the closed case is adequate for most applications, the reader may simply assume that G is closed for the remainder of these lectures.

Two G-structures $B \subset F^*(M)$ and $\tilde{B} \subset F^*(\tilde{M})$ are said to be *equivalent* if there exists a diffeomorphism $f : M \to \tilde{M}$ so that $f_1(B) = \tilde{B}$. The *equivalence problem* for G-structures is the problem of developing effective methods for determining whether or not two given G-structures are equivalent (and, if so, in how many ways). As I have already mentioned, it was Élie Cartan who first posed this general problem. He also proposed a method, nowadays known as the *equivalence method of É. Cartan*, for its solution.

Before discussing this method, I will illustrate its connections with geometry (and the geometry of PDE in particular) by the use of several examples of geometric structures that are effectively described in terms of G-structures.

EXAMPLE 1.1.1.1. Let $G = \mathrm{O}(n)$, the orthogonal group in n dimensions with respect to the standard inner product on $\mathbb{R}^n$. If M^n is endowed with a Riemannian metric g, then one can define

$$B_g = \{\ u \in F^*(M) \,|\, u : T_xM \to \mathbb{R}^n \text{ is an isometry } \}.$$

As the reader can verify, B_g is an $\mathrm{O}(n)$-structure on M. Conversely, if B is an $\mathrm{O}(n)$-structure on M, then there exists a unique Riemannian metric g_B on M defined by the rule $(g_B)_x(v, w) = u(v) \cdot u(w)$ for $v, w \in T_xM$ where u is any element of B_x. The very fact that B is an $\mathrm{O}(n)$-structure ensures that this does well-define g_B as a Riemannian metric on M. The two correspondences are inverse to each other, so a choice of a Riemannian metric is equivalent to a choice of $\mathrm{O}(n)$-structure.[2]

[2]This example generalizes directly to the pseudo-Riemannian case. One simply replaces $\mathrm{O}(n)$ by $\mathrm{O}(p, q)$.

The method of equivalence applied to $\mathrm{O}(n)$-structures will construct the Levi-Civita connection and the usual Riemannian curvature apparatus.

EXAMPLE 1.1.1.2. Suppose now that $n = 2m$ and let

$$J_m = \begin{pmatrix} 0 & \text{-}\mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix}$$

and define $G \subset \mathrm{GL}(2m, \mathbb{R})$ to be the subgroup of matrices that commute with J_m. As the reader can verify, one can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ in such a way that J_m becomes multiplication by i and G is thereby shown to be isomorphic to $\mathrm{GL}(m, \mathbb{C})$, so I will henceforth identify it as such.

Suppose now that J is an almost complex structure on a manifold M^{2m}, i.e., $J : TM \to TM$ is a bundle map satisfying $J^2 = \text{-Id}$. The uniqueness up to isomorphism of complex vector spaces of dimension m then implies that the set

$$B_J = \{\ u \in F_x^*(M) \mid u(J_x v) = J_m\, u(v) \text{ for all } v \in T_x M\ \}\,.$$

has the property that each fiber $\left(B_J\right)_x$ is a $\mathrm{GL}(m, \mathbb{C})$-orbit in F_x^*. Moreover, it is not difficult to show that when J is smooth, then so is B_J. Conversely, given a $\mathrm{GL}(m, \mathbb{C})$-structure $B \subset F^*(M)$, there is a unique almost complex structure J for which $B = B_J$. Thus, the two kinds of structure are equivalent.

The method of equivalence applied to $\mathrm{GL}(m, \mathbb{C})$-structures in this case will, as a first step, for each almost complex structure J, construct its Nijnhuis tensor N^J as a section of $TM \otimes \Lambda^2\left(T^*M\right)$ and show that it is a complete first order invariant. I.e., suppose that J and K are almost complex structures on $2m$-manifolds M and N respectively. Then for given points $x \in M$ and $y \in N$, there exists a local diffeomorphism $f : U \to N$, defined on an x-neighborhood U, that satisfies $f(x) = y$ and the condition that $f^*K\text{-}J$ vanishes to second order at x if and only if there exists a linear isomorphism $L : T_x M \to T_y N$ satisfying $L^*\left(K_y\right) = J_x$ and $L^*\left(N_y^K\right) N_x^J$. Moreover, the equivalence method will predict that K and J are locally equivalent if they satisfy $N^J = N^K = 0$. That this prediction is valid is the content of the Newlander-Nirenberg theorem.

EXAMPLE 1.1.1.3. Again, suppose that $n = 2m$ and let J_m be defined as in the previous example. Now, however, consider the subgroup $\mathrm{Sp}(m, \mathbb{R}) \subset \mathrm{GL}(2m, \mathbb{R})$ consisting of those matrices $A \in \mathrm{GL}(2m, \mathbb{R})$ that satisfy ${}^t\!A\, J_m\, A = J_m$. This group is known as the *symplectic group* of rank m and is a matrix group of dimension $2m^2{+}m$. Given a $\mathrm{Sp}(m, \mathbb{R})$-structure B on a $2m$-manifold M, one can define a non-degenerate, 2-form Ω on M by the rule

$$\Omega(v, w) = J_m\big(u(v)\big) \cdot u(w) \quad \text{for all } v, w \in T_x M,\ u \in B_x\,.$$

Conversely, the uniqueness up to isomorphism of symplectic vector spaces of a given dimension implies that any non-degenerate 2-form on M corresponds to a unique $\mathrm{Sp}(m, \mathbb{R})$-structure via this construction.

The method of equivalence in this case will show that $d\Omega$ is a complete first-order invariant of non-degenerate 2-forms, i.e., if Ω and Υ are non-degenerate 2-forms on $2m$-manifolds M and N respectively, then for given points $x \in M$ and $y \in N$, there exists a local diffeomorphism $f : U \to N$ where U is an x-neighborhood satisfying $f(x) = y$ and the condition that $f^*\Upsilon\text{-}\Omega$ vanishes to second order at x if and only if there exists a linear map $L : T_x M \to T_y N$

satisfying $L^*(\Upsilon_y) = \Omega_x$ and $L^*(d\Upsilon_y) = d\Omega_x$. Moreover, it will (correctly) predict that Ω and Υ are locally equivalent if they satisfy $d\Omega = d\Upsilon = 0$, i.e., that Darboux' Theorem holds.

EXAMPLE 1.1.1.4. Now suppose that $n = p + q$ where p and q are positive integers, and let $B_{p,q} \subset \mathrm{GL}(n, \mathbb{R})$ be the Borel subgroup

$$B_{p,q} = \left\{ \begin{pmatrix} A & 0 \\ L & B \end{pmatrix} \,\middle|\, A \in \mathrm{GL}(p, \mathbb{R}),\, B \in \mathrm{GL}(q, \mathbb{R}), \text{ and } C \in M_{q,p} \right\}$$

Note that $B_{p,q}$ is the subgroup that preserves the q-dimensional subspace $Q \subset \mathbb{R}^n$ consisting of those vectors whose first p coordinates are zero. It follows that a $B_{p.q}$-structure B on M^n determines a unique sub-bundle $E \subset TM$ of rank q by the requirement that $u(E_x) = Q$ for all $u \in B_x$. Conversely, given a sub-bundle $E \subset TM$ of rank q, there is a unique $B_{p,q}$-structure on M that consists of those coframes $u \in F^*_x$ that satisfy $u(E_x) = Q$.

The method of equivalence applied to $B_{p,q}$-structures will show how to construct from each q-plane field E on M a structure tensor δ_E on M that is a section of the bundle $TM/E \otimes \Lambda^2(E^*)$.[3] Moreover, δ_E will be shown to be a complete first order invariant and the method will predict the Frobenius theorem, i.e., that any two q-plane fields on n-manifolds with vanishing structure tensors are locally equivalent.

Further analysis in the case where δ_E is non-zero (which contains many examples important in control theory) depends on the algebraic structure of this map and can be rather involved, as the examples to be considered below will demonstrate.

EXAMPLE 1.1.1.5. More generally, let $I_0 \subset A^*(\mathbb{R}^n)$ be any graded ideal in the algebra of (constant coefficient) alternating forms on $\mathbb{R}^n$. Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be the group of linear transformations g whose induced action on $A^*(\mathbb{R}^n)$ preserves I_0. Then a G-structure B on M^n defines an ideal $I \subset \mathcal{A}^*(M)$ (which, of course, need not be differentially closed) by the requirement that a p-form ϕ on M belongs to I if and only if $(u^{-1})^*(\phi_x)$ lies in I_0 for all $x \in M$ and $u \in B_x$. Such ideals are sometimes said to be of *constant algebraic type* since for any two points $x, y \in M$, there exists a linear isomorphism $L : T_xM \to T_yM$ satisfying $L^*(I_y) = I_x$.

Conversely, given an ideal $I \subset \mathcal{A}^*(M)$ of constant algebraic type in this sense, once can clearly associate to it a G-structure where G is the group of automorphisms of a fixed representative I_0 of this type. As the examples to be presented below drawn from the study of partial differential equations indicate, the analysis via the method of equivalence of these sorts of G-structures turns out to be an effective method of studying the original PDE.

EXAMPLE 1.1.1.6. The triviality of this last example belies its importance, as will be seen. Suppose that $G = \{e\}$ is simply the identity matrix in $\mathrm{GL}(n, \mathbb{R})$. An $\{e\}$-structure on M is simply a submanifold $B \subset F^*(M)$ that intersects each fiber in one point and projects submersively (and hence, diffeomorphically) onto M. Hence, B is simply the image of a smooth global section of $F^*(M)$, i.e., a coframing $\eta = (\eta^i)$ of M. Thus, an $\{e\}$-structure can be identified with a global coframing of M.

[3]This tensor is implicit in the works of Élie Cartan, who used it extensively, as did many of his students. Modern authors sometimes refer to it as the Martinet tensor, but this appelation is not universal. Calling it the 'Cartan structure tensor' would be fair but hardly descriptive, given the plethora of objects that could be so designated.

1.1.2. *The tautological 1-form.* A distinguishing feature of the coframe bundle F^* that is inherited by all of its subbundles is the presence of a canonical 1-form with values in $\mathbb{R}^n$.

DEFINITION 1.1.2.1. For any G-stucture $B \subset F^*(M)$, the *tautological* 1-form ω is defined by

$$\omega(v) = u\big(\pi'(u)(v)\big) \qquad \text{for all } v \in T_u B.$$

Thus, for $u \in B$, the linear map ω_u is defined to be the composition

$$\begin{array}{ccc} T_u B & & \\ {\scriptstyle \pi'(u)}\big\downarrow & & \\ T_{\pi(u)}M & \xrightarrow{\ u\ } & \mathbb{R}^n, \end{array}$$

so that ω is a 1-form on B with values in $\mathbb{R}^n$. This 1-form seems to be known by various names in the literature. In some physics literature, for example, it is known as the 'soldering form'. Interest in this 1-form stems from its invariance properties, to which I now turn.

By its very construction, ω can be regarded as the pullback to B (via the inclusion map) of the tautological 1-form on F^* itself (thought of as the canonical $\mathrm{GL}(n,\mathbb{R})$-structure on M).

It is helpful to look at a formula for ω in a local trivialization. If η is a local section of B with domain $U \subset M$, let $H : U \times G \to B$ be the inverse trivialization defined earlier: $H(x, g) = g^{-1}\,\eta_x$. Unwinding the defintions yields the pullback formula

$$H^*(\omega) = g^{-1}\,\eta.$$

Thus, writing $\omega = \big(\omega^i\big)$, one sees that the n components of ω are linearly independent 1-forms whose simultaneous kernel consists of the vectors tangent to the π-fibers of B. In particular, ω is π-semi-basic. This description also makes it clear that ω has the 'reproducing property': $\eta^*(\omega) = \eta$ for any local section η of B.

The most important property of ω, however is the way it detects the prolongations of diffeomorphisms of the base manifold M.

PROPOSITION 1.1.2.2. *If $f : M_1 \to M_2$ is a diffeomorphism and $B_i \subset F^*(M_i)$ are G-structures satisfying $f_1\big(B_1\big) = B_2$, then $f_1^*(\omega_2) = \omega_1$. Conversely, if $U \subset B_1$ is an open subset of a G-structure on M_1 with the property that its π-fibers are connected and $\phi : U \to F^*(M_2)$ is any smooth mapping satisfying $\phi^*(\omega_2) = \omega_1$, then there exists a unique smooth mapping $f : \pi(U) \to M_2$ that satisfies $f \circ \pi_1 = \pi_2 \circ g$. Moreover, f is a local diffeomorphism and g is the restriction to U of f_1.*

PROOF. The first statement is just a matter of unwinding the definitions and applying the chain rule. One has a commutative diagram of maps:

$$\begin{array}{ccc} B_1 & \xrightarrow{\ f_1\ } & B_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\ f\ } & M_2 \end{array}$$

and, starting with a vector $v \in T_u B$, one has

$$\begin{aligned}
f_1^*(\omega_2)(v) &= \omega_2\bigl(f_1'(u)(v)\bigr)\\
&= f_1(u)\bigl(\pi_2'(f_1(u))(f_1'(u)(v))\bigr) = f_1(u)\bigl((\pi_2\circ f_1)(u)(v)\bigr)\\
&= f_1(u)\bigl((f\circ\pi_1)(u)(v)\bigr) = f_1(u)\bigl(f'(\pi_1(u))(\pi_1'(u)(v))\bigr)\\
&= u\bigl((f'(\pi_1(u)))^{-1} f'(\pi_1(u))(\pi_1'(u)(v))\bigr) = u\bigl((\pi_1'(u)(v))\bigr)\\
&= \omega_1(v).
\end{aligned}$$

The more interesting result is the converse. Suppose now that B is a G-structure on M, that $U \subset B$ is an open set with connected π-fibers, and that $g : U \to F^*(M_2)$ is a smooth map satisfying $g^*(\omega_2) = \omega_1$. Since, in each case, the kernel of ω_i is the tangent space to the π-fibers, it follows from the hypothesis that the π_1-fibers U_x of U are connected for all $x \in M$ that g must map each U_x into some π_2-fiber $F^*_{f(x)}(M_2)$. The map $f : \pi_1(U) \to N$ must be smooth since it can be written locally as a composition of the form $\pi_2 \circ g \circ \eta$ where η is a smooth local section of U over $\pi_1(U)$. Moreover, by construction, $f \circ \pi_1 = \pi_2 \circ g$. The equation $g^*(\omega_2) = \omega_1$ implies, in particular, that $\pi_2 \circ g$ is a submersion, so f must also be a submersion, and hence, for dimension reasons, a local diffeomorphism. Thus, the map f_1 is well-defined on U and, by the first part of the proof, must satisfy $f_1^*(\omega_2) = \omega_1$. Now since $\pi_2 \circ g = f \circ \pi_1 = \pi_2 \circ f_1$, it follows that there is a function $a : U \to \mathrm{GL}(n,\mathbb{R})$ so that $g(u) = f_1(u)\cdot a(u)$. However, using this formula for g and unwinding the definitions as before, one finds that

$$\omega_1 = g^*(\omega_2) = a^{-1}\, f_1^*(\omega_2) = a^{-1}\,\omega_1\,.$$

However, the independence of the n components of ω_1 now imply that a must be the map to the identity matrix. Thus, $g = f_1$, as desired. Finally, note that because f_1 commutes with the action of $\mathrm{GL}(n,\mathbb{R})$ and hence of G, it follows that g must do so as well, so that, at least locally, its image is an open subset of a G-structure on $f\bigl(\pi_1(U)\bigr)$

With the tautological 1-form in place, one can begin to see how the method of equivalence will go: To test whether or not two G-structures B_1 and B_2 are locally equivalent, one looks for integral manifolds of the 1-form $\theta = \omega_1 - \omega_2$ on the product manifold $B_1 \times B_2$. If one can find such an integral manifold $\Gamma \subset B_1 \times B_2$ that projects diffeomorphically onto each of the factors, then it will be the graph of a smooth map $g : B_1 \to B_2$ that satisfies $g^*(\omega_2) = \omega_1$ and hence, by the proposition just proved, will be induced by a diffeomorphism $f : M_1 \to M_2$ that induces an equivalence between the two G-structures. The reader familiar with Cartan's 'technique of the graph' from the theory of Lie groups will recognize this approach as a generalization of that technique.

The main difference (and difficulty) is that, for any matrix group G of positive dimension, the components of ω on a G-structure B do not form a coframing of B, which the usual form of the technique of the graph requires. In the following section, a method of completing ω to a coframing in a canonical way will be presented that works for many (indeed, most) matrix groups G. One can then proceed to the case of a manifold endowed with a global coframing, i.e., the case of an $\{e\}$-structure, the case treated in the section after that.

1.1.3. *Pseudoconnections and the intrinsic torsion.* In this section, it will often be useful to use the language of linear maps, homomorphisms, kernels, and cokernels in addition to the more pedestrian indicial notation, which, while often useful in calculations, tends to obscure the underlying concepts. For to this end, I will regard $\mathbb{R}^n$ as an abstract real vector space V of dimension n endowed with a basis $\mathbf{v}_i$ $(1 \le i \le n)$ that the reader should regard as the standard unit column vectors. The dual basis of V^* will be denoted by $\mathbf{v}^i$. For example, the tautological 1-form on $F^*(M)$ can be written in the form

$$\omega = \omega^i\,\mathbf{v}_i$$

where the ω^i are ordinary 1-forms on $F^*(M)$.

Let G be a matrix group and let $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ be its Lie algebra. I will suppose that $\dim G = s$ and, when necessary, I will let $\mathbf{u}_\alpha$ $(1 \le \alpha \le s)$ denote a basis of $\mathfrak{g}$. Because of the canonical inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(n,\mathbb{R}) = V \otimes V^*$, there are constants $u^i_{\alpha j}$ so that $\mathbf{u}_\alpha = u^i_{\alpha j}\,\mathbf{v}_i \otimes \mathbf{v}^j$, and these constants will sometimes be useful in formulae.

Now, one method of finding a coframing on a G-structure B goes as follows. Choose a connection form θ on B, i.e., a 1-form θ on B with values in $\mathfrak{g}$ with the following two properties:

(a) $\theta(X_v) = v$ for any $v \in \mathfrak{g}$ (here, X_v denotes the vector field on B whose flow Φ_v is defined by $\Phi_v(t,u) = u \cdot e^{vt}$).
(b) $R_a^*(\theta) = a^{-1}\theta a$ for all $a \in G$.

The existence of such a form follows from a standard partition of unity argument. Now write $\theta = \theta^\alpha\,\mathbf{u}_\alpha$. The s 1-forms θ^s then supplement the tautological 1-forms ω^i to define a global coframing on B.

The problem with this coframing is that, unlike the ω^i, there is no reason to expect the θ^α to be preserved by every equivalence between two G-structures. Evidently, if one is to find a canonical coframing of B, one must choose more carefully.

The key to making an informed choice in this situation is to start by examining the structure of the exterior derivative of the canonical 1-form in a local trivialization. Let $\eta = \eta^i\,\mathbf{v}_i$ be a local section of B with domain $U \subset M$. There are unique functions $C^i_{jk} = -C^i_{kj}$ on U so that

$$d\eta^i = \tfrac{1}{2}\,C^i_{jk}\,\eta^j \wedge \eta^k.$$

Writing $C = \frac{1}{2}\,C^i_{jk}\,\mathbf{v}_i \otimes \mathbf{v}^j \wedge \mathbf{v}^k$, and regarding C as a function from U to $V \otimes \Lambda^2(V^*)$, this can be written as a vector equation in the form

$$d\eta = \tfrac{1}{2}C\big(\eta \wedge \eta\big).$$

Consider the inverse trivialization $H : U \times G \to B$ associated to η as described in the previous subsection. For any connection θ on B, there exists a $\mathfrak{g}$-valued 1-form θ_0 on U so that $H^*(\theta) = g^{-1}dg + g^{-1}\theta_0 g$. Now, taking the exterior derivative of the relation $H^*(\omega) = g^{-1}\eta$ yields

$$\begin{aligned} H^*(d\omega) &= d\big(g^{-1}\,\eta\big) = -g^{-1}dg \wedge g^{-1}\eta + g^{-1}d\eta \\ &= -g^{-1}dg \wedge g^{-1}\eta + \tfrac{1}{2}g^{-1}C\big(\eta \wedge \eta\big) \\ &= H^*(-\theta \wedge \omega) + g^{-1}\,\big(\theta_0 \wedge \eta + \tfrac{1}{2}C\big(\eta \wedge \eta\big)\big) \\ &= H^*\,\big(-\theta \wedge \omega + \tfrac{1}{2}T\big(\omega \wedge \omega\big)\big) \end{aligned}$$

where $T = \frac{1}{2}\,T^i_{jk}\,\mathbf{v}_i \otimes \mathbf{v}^j \wedge \mathbf{v}^k$ is a function on B with values in the vector space $V \otimes \Lambda^2(V^*)$ that satisfies the equivariance $T(u \cdot A) = \rho_1\big(A^{-1}\big)\big(T(u)\big)$ where $\rho_1 = \rho_0 \otimes \Lambda^2(\rho_0^\dagger)$ is the induced representation on the stated tensor product. In other words, the *first structure equation* of Élie Cartan holds:

$$d\omega = -\theta \wedge \omega + \tfrac{1}{2} T(\omega \wedge \omega)$$

The function T is known as the *torsion function* of the connection θ. Of course, it represents a section of the vector bundle $TM \otimes \Lambda^2(T^*M)$ that is the associated bundle to B constructed from the G-representation ρ_1.

Now consider the effect on T of changing the connection. Let θ^* be any other connection on B. The difference $\theta^* - \theta$ is a $\mathfrak{g}$-valued 1-form on B that, by property (a) above, vanishes on vectors tangent to the fibers of $\pi : B \to M$. Hence, there exists a unique function $p : B \to \mathfrak{g} \otimes V^*$ so that

$$\theta^* = \theta + p(\omega).$$

In terms of the bases of $\mathfrak{g}$ and V^*, this function p can be written in the form $p = p^\alpha_i\, \mathbf{u}_\alpha \otimes \mathbf{v}^i$ for some functions p^α_i on B. Property (b) implies that p is G-equivariant, i.e., $p(u \cdot A) = (\mathrm{ad} \otimes \rho_0^\dagger)(A^{-1})\big(p(u)\big)$ for all $A \in G$. Conversely, for any G-equivariant $p : B \to \mathfrak{g} \otimes V^*$ and any connection 1-form θ, the formula $\theta^* = \theta + p(\omega)$ defines a connection 1-form on B.

Since

$$d\omega = -\theta \wedge \omega + \tfrac{1}{2} T(\omega \wedge \omega) = -\theta^* \wedge \omega + \tfrac{1}{2} T^*(\omega \wedge \omega),$$

where T^* is the torsion function associated to θ^*, it follows that

$$\tfrac{1}{2}(T^* - T)(\omega \wedge \omega) = (\theta^* - \theta) \wedge \omega = \big(p(\omega)\big) \wedge \omega = -\tfrac{1}{2}\delta(p)\,(\omega \wedge \omega),$$

i.e., that $T^* = T - \delta(p)$ where $\delta : \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2(V^*)$ is the G-equivariant linear map defined as the composition

$$\mathfrak{g} \otimes V^* \longrightarrow \big(V \otimes V^*\big) \otimes V^* \longrightarrow V \otimes \Lambda^2(V^*),$$

where the first map is the tensor product with V^* of the inclusion $\mathfrak{g} \hookrightarrow V \otimes V^*$ and the second map is skewsymmetrization in the second two factors.

The formula $T^* = T - \delta(p)$ suggests studying the kernel and cokernel of the map δ. For reasons that will be taken up again in a later section, these spaces have special notations and names:

$$\ker \delta = \mathfrak{g}^{(1)} \qquad \text{and} \qquad \operatorname{coker} \delta = H^{0,2}(\mathfrak{g}).$$

The space $\mathfrak{g}^{(1)}$ is known as the *first prolongation* of $\mathfrak{g}$ and the space $H^{0,2}(\mathfrak{g})$ is known as the *intrinsic torsion space* of $\mathfrak{g}$. This notation is somewhat misleading, since, as will be seen, these spaces depend not only on the abstract Lie algebra $\mathfrak{g}$ but its embedding into $\mathfrak{gl}(V) \simeq V \otimes V^*$. Because the map δ is G-equivariant, it folows that these two vector spaces have natural induced G-actions, i.e., there are representations $\rho^{(1)} : G \to \mathrm{GL}\big(\mathfrak{g}^{(1)}\big)$ and $\rho_{0,2} : G \to \mathrm{GL}\big(H^{0,2}(\mathfrak{g})\big)$.

For any element $t \in V \otimes \Lambda^2(V^*)$, let $[t] \in H^{0,2}(\mathfrak{g})$ denote its projection into the intrinsic torsion space. Then the computation above shows that $[T^*] = [T]$ as maps of B into $H^{0,2}(\mathfrak{g})$. In other words, the map $[T] : B \to H^{0,2}(\mathfrak{g})$ is independent of the choice of connection θ. This map $[T]$ is known as the *intrinsic torsion function* of the G-structure B. Because of the nature of the construction, this map is G-equivariant, i.e., for all $A \in G$ and $u \in B$,

$$[T](u \cdot A) = \rho_{0,2}(A^{-1})\big([T](u)\big).$$

PROPOSITION 1.1.3.1. *If $f : M_1 \to M_2$ is a diffeomorphism and $B_i \subset F^*(M_i)$ are G-structures satisfying $f_1(B_1) = B_2$, then $f_1^*([T_2]) = [T_1]$. Moreover, if $[T_1](u_1) = [T_2](u_2)$ for some $u_i \in B_i$, then there exists an open neighborhood $U \subset M_1$ of $x_1 = \pi(u_1)$ and a local diffeomorphism $f : U_1 \to M_2$ so that $f_1(u_1) = u_2$ and so that $f_1(B_1)$ is tangent to B_2 along the orbit $u_2 \cdot G$.*

PROOF. The first statement is easy to prove while the second is somewhat more subtle, though its significance should be clear: The intrinsic torsion is the only differential invariant of first order for G-structures.

To prove the first statement, suppose that θ_i is a connection 1-form on B_i and suppose that $f : M_1 \to M_2$ satisfies $f_1(B_1) = B_2$. Since f_1 is G-equivariant by constuction, it follows that $f_1^*(\theta_2)$ is a connection form on B_1, so that $f_1^*(\theta_2) = \theta_1 + p(\omega_1)$ for some $p : B_1 \to \mathfrak{g} \otimes V^*$. Consequently, it follows that $f_1^* T_2 = T_1 - \delta p$, so that

$$f_1^*([T_2]) = \big[f_1^*(T_2)\big] = \big[T_1 - \delta(p)\big] = [T_1],$$

as claimed.

To prove the second statement, it will be necessary to construct the desired map f. *I'll do this later.*

1.1.4. *e-structures.* The case of $\{e\}$-structures occupies a critical place in the theory, so it is worthwhile to devote some time to understanding this case. I will now collect together some of the basic results about $\{e\}$-structures that will be needed in the remainder of the lectures.

The first problem that I want to take up is how to compute the local and infinitesimal automorphisms of a given $\{e\}$-structure. I will begin the discussion by considering the construction of the differential invariants of such a structure.

Thus, suppose given a coframing $\omega = (\omega^i)$ on an n-manifold M. Since the ω^i are a basis for the 1-forms on M, it follows that there exist unique functions $C^i_{jk} = -C^i_{kj}$ on M so that

$$d\omega^i = \tfrac{1}{2} C^i_{jk}\, \omega^i \wedge \omega^j.$$

In general, of course, these functions will not be constants, but will satisfy differential equations got by taking the exterior derivative of the equations above. The result is equations of the form

$$\big(dC^i_{jk} + (C^i_{mk} C^m_{\ell j} - C^i_{mj} C^m_{\ell k})\, \omega^\ell\big) \wedge \omega^j \wedge \omega^k = 0$$

Regarding the case of global automorphisms, it is a theorem of Kobayashi [**Kob**] that, for any coframing η of a connected manifold M, the group Γ of (globally defined) smooth maps $f : M \to M$ that satisfy $f^*(\eta) = \eta$ can be given a smooth structure of a Lie group so that for each $x \in M$, the evaluation map $E_x : \Gamma \to M$ defined by $E_x(f) = f(x)$ is a smooth embedding of Γ as a closed submanifold of M.

Another important aspect of this problem is a uniqueness theorem, which should, roughly, state that two $\{e\}$-structures whose derived invariants are 'related in the same way' are, in fact, locally equivalent.

THEOREM. *Suppose that, on a domain $D \subset \mathbb{R}^s$, there are specified smooth functions $C^i_{jk} = -C^i_{kj}$ and F^α_i where the indices satisfy the ranges $1 \le i, j, k \le n$ and*

$1 \le \alpha \le s$. *Suppose further that there are n-manifolds M and N, endowed with coframings ω and η, respectively, so that there exist smooth mappings $a : M \to D$ and $b : N \to D$ satisfying*

$$d\omega^i = \tfrac{1}{2} C^i_{jk}\circ a\, \omega^j \wedge \omega^k \qquad d\eta^i = \tfrac{1}{2} C^i_{jk}\circ b\, \eta^j \wedge \eta^k$$
$$da^\alpha = F^\alpha_j \circ a\, \omega^j \qquad db^\alpha = F^\alpha_j \circ b\, \eta^j$$

Then if there exist $x \in M$ and $y \in N$ so that $a(x) = b(y)$, there exists an x-neighborhood $U \subset M$ and a smooth map $f : U \to N$ satisfying $f(x) = y$, $f^*(\eta) = \omega$, and $f^*(b) = a$.

Proof. Of course, the proof would like to use the Frobenius Theorem: On $M \times N$, one should consider the Pfaffian system I generated by the 1-forms $\theta^i = \eta^i - \omega^i$ restricted to the submanifold $Z \subset M \times N$ defined by $b - a = 0$. The difficulty is that one does not know anything a priori about the rank of the functions a and b, so there is no reason to believe that Z is anything like a smooth manifold. Moreover, even if it could be shown to be a smooth manifold, there is no reason to believe that the Pfaffian system I restricted to Z need have constant rank, making application of the Frobenius Theorem problematic. Thus, an alternative is needed. For this reason, I will first prove a lemma generalizing the Frobenius theorem that is of interest in its own right.

Lemma. *Let $\mathcal{I}$ be a differentially closed ideal on a manifold M of dimension $n+p$ and suppose that $\mathcal{I}$ is generated algebraically by a finite number of functions $\{z^1, \dots, z^s\}$ together with p 1-forms $\{\theta^1, \dots, \theta^p\}$ that are linearly independent. Then the set $Z = \{\, x \in M \,|\, z^\alpha(x) = 0 \text{for } 1 \le \alpha \le s \,\}$ is a disjoint union of n-dimensional integral manifolds of $\mathcal{I}$.*

Proof. By the usual uniqueness theorems in ordinary differential equations, it suffices to show that every point of Z lies in at least one n-dimensional integral manifold of $\mathcal{I}$ since it is clear that there is at most one n-dimensional integral manifold of $\mathcal{I}$ passing through each point of M.

Fix $x \in Z$. Note that, by the differential closure of $\mathcal{I}$ and using the index ranges $1 \le a, b \le n$, $1 \le \alpha, \beta \le s$, there must exist functions f^α_a, 1-forms ψ^α_β and ϕ^a_b, and 2-forms Υ^a_β on M so that

$$dz^\alpha = z^\beta\, \psi^\alpha_\beta + f^\alpha_b\, \theta^b$$
$$d\theta^a = z^\beta\, \Upsilon^a_\beta + \phi^a_b \wedge \theta^b$$

I claim that it follows that any integral curve of the θ^i that intersects Z must lie entirely in Z. For suppose that $\gamma : [0,1] \to M$ satisfies $\gamma^*(\theta^i) = 0$ and $\gamma(0) = z \in Z$. Then the functions ζ^α on $[0,1]$ defined by $\zeta^\alpha(t) = z^\alpha\big(\gamma(t)\big)$ satisfy the initial conditions $\zeta^\alpha(0) = 0$ and the linear system of differential equations

$$\frac{d\zeta^\alpha}{dt} = \psi^\alpha_\beta\big(\gamma'(t)\big)\zeta^\beta,$$

which, by uniqueness, forces $\zeta^\alpha(t) = 0$ for all t.

Now, from the linear independence of the θ^a, it follows that there is a neighborhood of $z \in Z$ on which there exist vector fields X_i $(1 \le i \le n)$ that are linearly independent and satisfy $\theta^a(X_i) = 0$ for all a and i. It then follows that there exists a smooth map L from a cubic neighborhood of $0 \in \mathbb{R}^n$ to M that satisfies

$$L(t^1, \dots, t^n) = \exp_{t^n X_n} \circ \cdots \circ \exp_{t^1 X_1}(z).$$

By shrinking the neighborhood of 0, I can arrange that L is a smooth embedding of the neighborhood into M. The fundamental property enjoyed by L aside from satisfying $L(0) = z$ is that every curve of the form

$$\gamma(t) = L(x^1, \dots, x^{i-1}, t, 0, \dots, 0)$$

is an integral curve of X_i and hence of the 1-forms θ^a. In particular, it follows that the image of L lies entirely in the locus Z. It remains to show that L is an integral manifold of the θ^a. To see this, set $\eta^a = L^*(\theta^a)$ and $\varphi^a_b = L^*(\phi^a_b)$. Then the structure equations above show that

$$d\eta^a = \varphi^a_b \wedge \eta^b.$$

The η^a also have the property that they vanish along curves of the form $t \mapsto (x^1, \dots, x^{i-1}, t, 0, \dots, 0)$, i.e., when one writes $\eta^a = A^a_i, dx^i$, the functions A^a_i satisfy

$$A^a_i(x^1, \dots, x^i, 0, \dots, 0) = 0$$

and, by the equations for $d\eta^a$, there are equations of the form

$$\frac{\partial A^a_i}{\partial x^j} - \frac{\partial A^a_j}{\partial x^i} = B^a_{bi} A^b_j - B^a_{bj} A^b_i\,.$$

Now the proof of the usual Frobenius Theorem applies: uniqueness in a succession of Cauchy problems shows that the A^a_i must vanish identically. Hence $0 = \eta^a = L^*(\theta)$, so that the image of L is an integral manifold of the θ^a, as desired. □

Using this Lemma, the proof of the Theorem will be straightforward once the appropriate ideal has been constructed. First, let $a(x) = b(y) = a_0 \in D$. By a theorem of Whitney [**GG**], in a neighborhood of (a_0, a_0) in $D \times D$, there exist smooth functions $F^\alpha_{i\beta}$ and $C^i_{jk\beta}$ so that

$$F^\alpha_i(p) - F^\alpha_i(q) = F^\alpha_{i\beta}(p,q)(p^\beta - q^\beta)$$
$$C^i_{jk}(p) - C^i_{jk}(q) = C^i_{jk\beta}(p,q)(p^\beta - q^\beta)$$

for all (p,q) in this neighborhood. It follows that if we define functions $z^\alpha = a^\alpha - b^\alpha$ on $M \times N$, then there exist functions $H^\alpha_{i\beta}$ and $G^i_{jk\beta}$ on a neighborhood W of $(x,y) \in M \times N$ so that

$$F^\alpha_i \circ a - F^\alpha_i \circ b = H^\alpha_{i\beta} z^\beta$$
$$C^i_{jk} \circ a - C^i_{jk} \circ b = G^i_{jk\beta} z^\beta$$

By the given structure equations, one then gets a formula

$$dz^\alpha = d(a^\alpha - b^\alpha) = F^\alpha_i \circ a\, \omega^i - F^\alpha_i \circ a\, \eta^i = H^\alpha_{i\beta} z^\beta\, \omega^i - F^\alpha_i \circ b\, \theta^i.$$

Also, by the given structure equations

$$\begin{aligned} d\theta^i = d\eta^i - d\omega^i &= \tfrac{1}{2} C^i_{jk} \circ b\, \eta^i \wedge \eta^j - \tfrac{1}{2} C^i_{jk} \circ a\, \omega^i \wedge \omega^j \\ &= \tfrac{1}{2} C^i_{jk} \circ b \left(\eta^i \wedge \eta^j - \omega^i \wedge \omega^j\right) - \tfrac{1}{2} C^i_{jk\beta} z^\beta\, \omega^i \wedge \omega^j \\ &= \tfrac{1}{2} C^i_{jk} \circ b \left(\theta^i \wedge \theta^j + \theta^i \wedge \omega^j + \omega^i \wedge \theta^j\right) - \tfrac{1}{2} G^i_{jk\beta}\, z^\beta\, \omega^i \wedge \omega^j. \end{aligned}$$

Thus, the ideal $\mathcal{I}$ generated by the θ^i and the z^α satisfies the hypotheses of the Lemma in a neighborhood of $(x,y) \in M \times N$. An application of the Lemma then gives the existence of an n-dimensional integral manifold of $\mathcal{I}$ in a neighborhood of (x,y). By construction, the 1-forms ω^i are linearly independent on this integral manifold, so near (x, y) it is the graph of the desired map f. $\square$

So much for uniqueness. However, for many applications, it is important to know how many coframings exist (up to diffeomorphism) where the derived invariants satisfy some given constraints in advance. The typical case occurs in geometric problems where the calculations have led to some formulae of the form

$$d\omega^i = \tfrac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k,$$
$$da^\alpha = F^\alpha_j\, ,\omega^j.$$

where the functions C^i_{jk} and F^α_i are explicitly known in terms of the functions a. Furthermore, exterior differentiation of these equations produces no new relations among the ω^i and the a^α. One would then like to know whether there exists such a system or not.

The simplest case of this kind of question is when there are no differential invariants a^α, i.e., when the C^i_{jk} are constants. In this case, the exterior derivative of the equation $d\omega^i = \frac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k$ is the equation

$$0 = d\big(d\omega^i\big) = \big(C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk}\big)\, \omega^j \wedge \omega^k \wedge \omega^l,$$

so it follows that such a system with the ω^i independent cannot exist unless the constants C^i_{jk} satisfy the Jacobi identities:

$$0 = C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk}\, .$$

for all i, j, k, and l. As Lie proved in his Third Fundamental Theorem, this necessary condition is also sufficient.

In the more general case where C^i_{jk} is allowed to depend on some parameters and their derivatives in terms of the coframing are specified, a generalization of the Jacobi identity is required. This generalized condition is found the same way as the Jacobi condition: one computes the exterior derivatives of the given equations and notes the identities that the functions C and F must satisfy in order for the exterior derivatives to be consequences of the equations themselves. According to the following theorem, these necessary identities are also sufficient in the case where the C and F are analytic functions of a. Thus, this is a generalization of Lie's Third Fundamental Theorem. It is due, in this form, to Élie Cartan.

THEOREM. *Suppose that $D \subset \mathbb{R}^s$ is an open set on which there exist real analytic functions $C^i_{jk} = -C^i_{kj}$ and F^α_i (where the index ranges are $1 \le i, j, k \le n$ and $1 \le \alpha, \beta, \gamma \le s$) and suppose that these functions satisfy the equations*

$$\frac{1}{2}\left(F^\alpha_j \frac{\partial C^i_{kl}}{\partial a^\alpha} + F^\alpha_k \frac{\partial C^i_{lj}}{\partial a^\alpha} + F^\alpha_l \frac{\partial C^i_{jk}}{\partial a^\alpha} \right) = C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk}$$

and

$$F^\beta_i \frac{\partial F^\alpha_j}{\partial a^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial a^\beta} = F^\alpha_l C^l_{ij}\, .$$

Then for every $a_0 \in D$, there exists a real analytic n-manifold M together with a real analytic coframing $\eta = (\eta^i)$ and a real analytic mapping $a : M \to D$ satisfying the equations

$$
\begin{aligned}
d\omega^i &= \tfrac{1}{2} C^i_{jk}\circ a\, \omega^j \wedge \omega^k,\\
da^\alpha &= F^\alpha_j \circ a\,, \omega^j.
\end{aligned}
$$

REMARK. Of course by the previous theorem, this manifold and coframing are unique up to diffeomorphism. Note that a need not have constant rank or even have rank n on a dense open subset of M. An example in [**Br?**] that classifies the torsion-free connections on 4-manifolds with holonomy conjugate to the irreducible degree 4 representation of $\mathrm{SL}(2,\mathbb{R})$ displays how complicated the maps a and their images can be.

PROOF. The proof is a straightforward application of the Cartan-Kähler theorem. Set $X = \mathbb{R}^n \times \mathrm{GL}(n,\mathbb{R}) \times D$ with projections to the factors given by $x : X \to \mathbb{R}^n$, $g : X \to \mathrm{GL}(n,\mathbb{R})$, and $a : X \to D$. Define the $\mathbb{R}^n$-valued 1-form $\omega = (\omega^i)$ on X by the formula $\omega = p^{-1}\, dx$. Now define the 1-forms and 2-forms

$$
\begin{aligned}
\theta^\alpha &= da^\alpha - F^\alpha_j\, \omega^j\\
\Theta^i &= d\omega^i - \tfrac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k
\end{aligned}
$$

Let $\mathcal{I}$ be the differential ideal generated by the 1-forms θ^α and the 2-forms Θ^i. Then, because of the assumptions that we made about the functions C^i_{jk} and F^α_i, a computation reveals that

$$
\begin{aligned}
d\theta^\alpha &= -\frac{\partial F^\alpha_i}{\partial a^\beta}\,\theta^\beta - F^\alpha_i \Theta^i\\
d\Theta^i &= -C^i_{jk}\,\Theta^j \wedge \omega^k - \tfrac{1}{2}\frac{\partial C^i_{jk}}{\partial a^\beta}\,\theta^\beta \wedge \omega^j \wedge \omega^k
\end{aligned}
$$

In particular, $\mathcal{I}$ is differentially generated by the given 1-forms and 2-forms.

Now, from the definition of ω^i, it follows that there exist 1-forms

$$\gamma^i_j \equiv (g^{-1}\, dg)^i_j \mod \{\omega^1, \dots, \omega^n\}$$

so that $\Theta^i = -\gamma^i_j \wedge \omega^j$. Hence, the equations $\theta^\alpha = \gamma^i_j = 0$ define a field of n-dimensional integral elements of $\mathcal{I}$. One easily establishes (using any flag) that the characters of these elements are $s_0 = s$, and $s_i = n$ for $1 \leq i \leq n$. Moreover, the space of integral elements at any point satisfing the independence condition $\omega^1 \wedge \cdots \omega^n \neq 0$ is parametrized by $n\binom{n+1}{2}$ parameters $p^i_{jk} = p^i_{kj}$ by the equations

$$\theta^\alpha = \gamma^i_j - p^i_{jk}\,\omega^k = 0.$$

Thus, $S = n\binom{n+1}{2} = s_1 + 2s_2 + \cdots + n\, s_n$ and Cartan's Test is verified, so that all of these integral elements are ordinary. By the Cartan-Kähler Theorem, there exist n-dimensional integral manifolds satisfying the independence condition $\omega^1 \wedge \cdots \omega^n \neq 0$ through every point of X. Pulling back the 1-forms ω^i and the coordinate projection a to such an integral manifold M passing through $(0, I_n, a_0) \in X = \mathbb{R}^n \times \mathrm{GL}(n,\mathbb{R}) \times D$ then produces the desired coframing and mapping. $\square$

REMARK 1. The reader may have noticed that very little about the structure of the domain D is used in the proof. In actual fact, the theorem can be stated without reference to coordinates on D and, in this form, it is perhaps clearer, though not as computationally immediate. It does sometimes happen that this version is useful, however, so I will insert a short discussion of this here: One can avoid any mention of indices or coordinates by resorting to the following language: Let V be a vector space of dimension n and let D be a (real analytic) manifold of some dimension s. Suppose given a (real analytic) function $C : D \to V \otimes \Lambda^2(V^*)$ and a (real analytic) bundle map $F : D \times V \to TD$ (which is not assumed to be of constant rank). Assume also that C and F satisfy certain natural PDE of the form

$$\{F, C\}_1 = J(C)$$

$$\{F, F\}_2 = \langle\, F,\, C\,\rangle$$

(I leave the definition of the various operators to the interested reader.) Then for every $a_0 \in D$, there exist a neighborhood M of $0 \in \mathbb{R}^n$ on which there exist a V-valued coframing ω and a (real analytic) function $a : M \to D$ satisfying $a(0) = a_0$ and the equations $d\omega = C{\circ}a\,(\omega{\wedge}\omega)$ and $da = F{\circ}a\,(\omega)$. (Here, 'da' is to be interpreted as the differential $da : TM \to TD$.)

Remark 2. The hypothesis of real analyticity is probably not necessary and a generalization of the usual argument for Lie's Third Fundamental Theorem using only ODE could be constructed, but I leave this also for the interested reader. The real analytic case more than suffices for all that I have in mind.

Remark 3. Let $D^* \subset D$ be the open, dense subset consisting of those $a_0 \in D$ where $F(a_0)$, considerd as an s-by-n matrix, has maximal rank $r \le \min(s, n)$. Then D^* is foliated by the images of maps $a : M \to D$ associated to coframings of this type, with leaves of dimension r and hence of codimension $s{-}r$. Thus, it makes sense to say that the 'generic' coframing of this type depends on $s{-}r$ parameters and has an infinitesimal symmetry algebra of dimension $n{-}r$. However, this statement can be deceptive and so must be handled with care. For example, it can happen that there is a stratum $D_\alpha \subset D$ where the rank of F is some $r_\alpha < r$ that is foliated by images of maps a and the codimension of this foliation is *greater* than $s - r$. In this case, the moduli space of 'special' solutions associated to D_α will have greater 'dimension' than that of the moduli space of 'generic' solutions.

1.2. The crude method.

1.2.1. *The first structure equation.*

1.2.2. *Prolongation.*

1.2.3. *The tower of bundles.*

1.2.4. *Polynomial differential invariants.*

1.3. Reduction.

1.3.1. *The torsion representation.*

1.3.2. *Stabilizer types and reduction.*

2. Elementary Examples

2.1. Riemannian geometry.

2.2. 3-webs in the plane.

2.3. .

3. Advanced Examples

3.1. **Real Lagrangian bipolarizations.** This example will treat the geometry of an $\mathrm{Sp}(n,\mathbb{R})$-structure and two substructures, the substructure preserving a Lagrangian foliation and the substructure preserving a Lagrangian bi-polarization.

3.2. **CR-hypersurfaces in $\mathbb{C}^2$.** In this example, I will give an exposition of Cartan's solution of the equivalence problem for non-degenerate hypersurfaces in complex 2-manifolds. Of course, the theory has been extensively developed in the intervening years, with the general solution for a non-degenerate hypersurface in a complex n-manifold being the subject of a famous paper by Chern and Moser [**ChMo**] as well of several works by Tanaka and his school [**Ta??**].

3.2.1. *The geometric problem and its G-structure.* Suppose that $M^3 \subset X$ is a smooth real hypersurface in a complex 2-manifold X, which can be taken to be $\mathbb{C}^2$ if desired . For each $x \in M$, the tangent plane T_xM cannot be a complex subspace of T_xX, but contains a unique complex subspace $D_x \subset T_xM$ of complex dimension 1. Thus, M inherits a geometric structure from being immersed as a hypersurface in a complex 2-manifold.

DEFINITION 3.2.1.1. A (smooth) CR-*structure* on a 3-manifold M is a choice of a (smooth) rank 2 subbundle $D \subset TM$ together with a choice of complex structure on D, i.e., a smooth bundle map $J : D \to D$ satisfying $J^2 = -\mathrm{Id}_D$.

In the real analytic category, every CR-structure on a 3-manifold is locally induced by an immersion into $\mathbb{C}^2$.

PROPOSITION 3.2.2. *Let (D, J) be a real analytic* CR-*structure on M^3. Then for each point $x \in M$ there exists an x-neighborhood U and a real analytic embedding $Z : U \to \mathbb{C}^2$ so that (D, J) is the* CR-*structure on U induced by the embedding Z.*

PROOF. On a neighborhood U of x choose a real analytic, non-vanishing real 1-form ρ that annihilates D and a real-analytic, complex valued 1-form η linearly independent from ρ that satisfies $\eta(Jv) = \imath\,\eta(v)$ for all $v \in D$. Then any complex-valued 1-form ζ on U that satisfies $\zeta(Jv) = \imath\,\zeta(v)$ is a linear combination of ρ and η. As the reader can check, to construct the desired Z, one must find two complex functions z^1 and z^2 in a neighborhood of x whose differentials are linearly independent and that satisfy $dz^k(Jv) = \imath\,dz^k(v)$, i.e., so that $dz^k{\wedge}\rho{\wedge}\omega = 0$. Now, on $N = U \times \mathbb{C}$ with second projection $z : N \to \mathbb{C}$, let $\mathcal{I}$ be the ideal generated by the two 3-forms that are the real and imaginary parts of $dz{\wedge}\rho{\wedge}\omega$. The characters are $s_i = 0$ for $i \neq 2$ and $s_2 = 2$. The space of 3-dimensional integral elements that satisfy the independence condition $\rho{\wedge}\omega{\wedge}\bar{\omega} \neq 0$ is clearly of dimension 4, so the system is in involution. Choose two integral manifolds Σ_i, $i = 1, 2$ of this system that pass through $(x, 0) \in N$ but that are not tangent there. Each is then the graph of a function z^i that satisfies $dz^k{\wedge}\rho{\wedge}\omega = 0$ and the condition that the two integral manifolds not be tangent is equivalent to $dz^1{\wedge}dz^2 \neq 0$. $\square$

REMARK. The famous Levy-Nirenberg example shows that the assumption of real analyticity is necessary here.

Suppose now that M^3 is endowed with a CR-structure (D, J). Let $V = \mathbb{R} \oplus \mathbb{C}$ and think of V as the space of columns of height 2 whose first entry is real and whose second entry is complex. A coframe $u : T_xM \to V$ will be said to be 0-*adapted* to (D, J) if $u(D_x) = \mathbb{C} \subset V$ and, moreover, $u(Jv) = i\,u(v)$ for all $v \in D_x$. I will let $B_0 \subset F^*(M, V)$ denote the space of 0-adapted V-valued coframes on M. If u and u^* lie in B_0 and share the same basepoint, then

$$u^* = \begin{pmatrix} r & 0 \\ b & a \end{pmatrix} u$$

where r is a real number and a and b are complex, with $a \neq 0$. Thus, B_0 is a G_0-structure on M where

$$G_0 = \left\{ \begin{pmatrix} r & 0 \\ b & a \end{pmatrix} \middle| \; r \in \mathbb{R}^*,\ a \in \mathbb{C}^*,\ \text{and } b \in \mathbb{C} \right\}.$$

Conversely, given a G_0-structure B_0 on M, there is canonically associated to it a unique CR-structure (D, J) that gives rise to it via this construction. Thus, the two sorts of structures are equivalent.

3.2.2. *The first analysis.* Now let B_0 be a G_0-structure on M^3. I will write the canonical V-valued 1-form ω on B_0 in the form

$$\omega = \begin{pmatrix} \theta \\ \eta \end{pmatrix}$$

where θ is a real-valued 1-form and η is a complex-valued 1-form. The first structure equation can be written in the form

$$d\begin{pmatrix} \theta \\ \eta \end{pmatrix} = -\begin{pmatrix} \rho_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \theta{\wedge}\big(b\,\eta + \bar{b}\,\bar{\eta}\big) + \imath L\,\eta{\wedge}\bar{\eta} \\ \theta{\wedge}\big(c\,\eta + e\,\bar{\eta}\big) + T\,\eta{\wedge}\bar{\eta} \end{pmatrix}$$

where L is a real function on B_0 but the other coefficients are allowed to be complex. Clearly, by adding multiples of θ, η and $\bar{\eta}$ to the pseudo-connection forms ρ_0, α_0, and β_0, I can arrange that $b = c = e = T = 0$, but I cannot affect L. Thus, I can assume that the structure equations have the form

$$d\begin{pmatrix} \theta \\ \eta \end{pmatrix} = -\begin{pmatrix} \rho_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \imath L\,\eta{\wedge}\bar{\eta} \\ 0 \end{pmatrix}.$$

Differentiating the first equation $d\theta = -\rho_0{\wedge}\theta + \imath L\,\eta{\wedge}\bar{\eta}$ and reducing modulo θ gives the relation

$$dL \equiv L\big(\alpha_0 + \bar{\alpha}_0 - \rho_0\big) \mod \theta, \eta, \bar{\eta},$$

so it follows that either L vanishes identically on a fiber of B_0 or is nowhere zero there.

The case where L vanishes identically, i.e., the intrinsic torsion of the G_0-structure vanishes turns out not to be very interesting. In this case, one can calculate that the characters of the Lie algebra $\mathfrak{g}_0$ are $s_1 = 3$, $s_2 = 1$, and $s_3 = 0$. Moreover the variability of the pseudo-connection is of dimension $5 = s_1 + 2s_2$, so G_0 is semi-involutive and all of the real-analytic G_0-structures with vanishing torsion are equivalent. Thus, it makes sense to concentrate on the (generic) case where L is nowhere vanishing.

Now, there is a direct geometric interpretation of L. Since θ is a non-zero multiple of $\pi^*(\sigma)$ where σ is any non-vanishing 1-form with $D = \ker\sigma$, it follows that $\theta{\wedge}d\theta = \imath L\,\theta{\wedge}\eta{\wedge}\bar{\eta}$ is non-zero if and only if $\sigma{\wedge}d\sigma$ is non-zero, i.e., if and only if D is a contact plane field on M^3.

DEFINITION 3.2.2.1. A CR-structure (D, J) on M^3 is *non-degenerate* if D is nowhere-integrable, i.e., is a contact structure on M.

Thus, the condition that L be nowhere vanishing is the condition that the original CR-structure be non-degenerate. From now on, I am going to assume that this is the case. This assumption leads directly to the first reduction: Set

$$B_1 = \{\ u \in B_0 | L(u) = 1\ \}.$$

Then B_1 is a G_1-structure on M where

$$G_1 = \left\{ \begin{pmatrix} a\bar{a} & 0 \\ b & a \end{pmatrix} \middle|\ a \in \mathbb{C}^* \text{ and } b \in \mathbb{C} \right\}.$$

Pulling all of the forms on B_0 back to B_1 and giving them the same names, the structure equations on B_1 now read

$$d\begin{pmatrix} \theta \\ \eta \end{pmatrix} = -\begin{pmatrix} \alpha_0 + \bar{\alpha}_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} (\alpha_0 + \bar{\alpha}_0 - \rho_0) \wedge \theta + \imath\, \eta \wedge \bar{\eta} \\ 0 \end{pmatrix}.$$

where $\alpha_0 + \bar{\alpha}_0 - \rho_0 = a\theta + b\eta + \bar{b}\bar{\eta}$ for some functions a and b on B_1. Subtracting $b\eta$ from α_0 reduces the function b to zero and the structure equations become

$$d\begin{pmatrix} \theta \\ \eta \end{pmatrix} = -\begin{pmatrix} \alpha_0 + \bar{\alpha}_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \imath\, \eta \wedge \bar{\eta} \\ 0 \end{pmatrix}.$$

Now the torsion is constant. If the algebra $\mathfrak{g}_1$ were involutive, then reaching this point would imply that any two non-degenerate G_0-structures were locally equivalent. However, one easily computes that the characters of this algebra are $s_1 = 3$, $s_2 = 1$, and $s_3 = 0$ while the pseudo-connections with this torsion are determined up to a replacement of the form $(\alpha_0, \beta_0) \mapsto (\alpha_0^*, \beta_0^*)$ where

$$\begin{pmatrix} \alpha_0^* \\ \beta_0^* \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} s^1 & 0 \\ s^2 & s^1 \end{pmatrix} \begin{pmatrix} \theta \\ \eta \end{pmatrix},$$

and s^1 and s^2 are arbitrary complex-valued functions on B_1, so $\dim \mathfrak{g}_1^{(1)} = 4 < s_1 + 2s_2 + 3s_3 = 5$. Hence, there remains the possibility that there will be differential invariants at some higher order.

3.2.3. *Prolongation and further reductions.* According to the prescription of the method of equivalence, I now construct a $\mathfrak{g}_1^{(1)}$-bundle $B_1^{(1)}$ over B_1 that consists of the coframes on B_1 with values in $V \oplus \mathfrak{g}_1$ that satisfy the structure equations of B_1. For simplicity, I will identify $V \oplus \mathfrak{g}_1$ with $\mathbb{R} \oplus \mathbb{C}^3$ thought of as the columns of height 4 with the first entry real and the remaining three complex. In the trivialization $B_1^{(1)} = B_1 \times \mathfrak{g}_1^{(1)}$ induced by the section $B_1 \to B_1^{(1)}$ represented by a choice of α_0 and β_0 on B_1 as above, the canonical 1-form $\omega^{(1)}$ has the form

$$\omega^{(1)} = \begin{pmatrix} \theta \\ \eta \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s^1 & 0 & 0 & 0 \\ s^2 & s^1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \theta \\ \eta \\ \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} \theta \\ \eta \\ \alpha_0 - s^1\theta \\ \beta_0 - s^2\theta - s^1\eta \end{pmatrix}$$

where, of course, the functions s^1 and s^2 now represent coordinates on $\mathfrak{g}_1^{(1)}$ and so are independent from the functions on B_1. The structure equations on $B_1^{(1)}$ have the form:

$$d\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\ 0&0&0&0\\ \sigma_0^1&0&0&0\\ \sigma_0^2&\sigma_0^1&0&0\end{pmatrix}\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha)\wedge\theta + \imath\,\eta\wedge\bar\eta\\ -\beta\wedge\theta-\alpha\wedge\eta\\ T_\alpha\\ T_\beta\end{pmatrix}$$

where T_α and T_β represent the torsion terms associated to those components of the canonical 1-form while σ_0^1 and σ_0^2 are 1-forms that satisfy $\sigma_0^i \equiv ds^i$ modulo semi-basic forms for the projection $B_1^{(1)} \to B_1$ but that are otherwise arbitrary.

Computing the exterior derivatives of the first two structure equations yields

$$\begin{aligned} 0 = d(d\theta) &= \quad -\left(T_\alpha + \overline{T_\alpha} - \imath\,\beta\wedge\bar\eta + \imath\,\bar\beta\wedge\eta\right)\wedge\theta \\ 0 = d(d\eta) &= -\left(T_\beta + \beta\wedge\bar\alpha\right)\wedge\theta - \left(T_\alpha + \imath\,\beta\wedge\bar\eta\right)\wedge\eta \end{aligned}$$

Setting $T_\alpha^* = T_\alpha + \imath\,\beta\wedge\bar\eta + 2\imath\,\bar\beta\wedge\eta$ and $T_\beta^* = T_\beta + \beta\wedge\bar\alpha$, these equations can be written in the form

$$\left(T_\alpha^* + \overline{T_\alpha^*}\right)\wedge\theta = \left(T_\beta^*\right)\wedge\theta + \left(T_\alpha^*\right)\wedge\eta = 0$$

and the second of these equations implies, via Cartan's Lemma, that there exist 1-forms ψ_1, ψ_2, and ψ_3 so that

$$\begin{aligned} T_\alpha^* &= \psi_2\wedge\theta + \psi_1\wedge\eta \\ T_\beta^* &= \psi_3\wedge\theta + \psi_2\wedge\eta \end{aligned}$$

Since T_α^* and T_β^* are semi-basic, it follows that the ψ_i must be also. Thus, by subtracting ψ_2 from σ_0^1 and ψ_3 from σ_0^2, I can suppose that $\psi_2 = \psi_3 = 0$. Then the remaining equation on T_α^* implies that

$$\left(\psi_1\wedge\eta + \overline{\psi_1\wedge\eta}\right)\wedge\theta = 0,$$

which implies that $\psi_1\wedge\eta = b\,\eta\wedge\theta + R\,\eta\wedge\bar\eta$ where b is a complex function and R is a real function. By adding $b\,\eta$ to σ_0^1, I can assume that $b = 0$, so that the structure equations now take the form

$$d\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\ 0&0&0&0\\ \sigma_0^1&0&0&0\\ \sigma_0^2&\sigma_0^1&0&0\end{pmatrix}\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha)\wedge\theta + \imath\,\eta\wedge\bar\eta\\ -\beta\wedge\theta-\alpha\wedge\eta\\ -\imath\,\beta\wedge\bar\eta - 2\imath\,\bar\beta\wedge\eta + R\,\eta\wedge\bar\eta\\ -\beta\wedge\bar\alpha\end{pmatrix}.$$

Now computing the exterior derivative of the $d\alpha$ equation modulo θ yields

$$0 = d(d\alpha) \equiv \left(dR - (\alpha+\bar\alpha)R - 2\imath(\overline{\sigma_0^1} - \sigma_0^1)\ \right)\wedge\eta\wedge\bar\eta \mod \theta$$

which implies

$$dR \equiv (\alpha+\bar\alpha)R + 2\imath(\overline{\sigma_0^1} - \sigma_0^1) \mod \theta,\eta,\bar\eta.$$

In particular, on each fiber of $B_1^{(1)} \to B_1$, the relation $dR = 2\imath\,d(\overline{s^1} - s^1)$ holds. It follows that the equation $R = 0$ defines a G_2-structure $B_2 \subset B_1^{(1)}$ on B_1 where G_2 is the subgroup consisting of those matrices in $\mathfrak{g}_1^{(1)}$ for which s^1 is real. I will now pull back all of the forms and functions on $B_1^{(1)}$ to B_2, write $\sigma_0^1 = \sigma_0 + \imath\,\tau$ where σ_0 and τ are real 1-forms, and write the structure equations on B_2 in the form

$$d\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\ 0&0&0&0\\ \sigma_0&0&0&0\\ \sigma_0^2&\sigma_0&0&0\end{pmatrix}\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha){\wedge}\theta + \imath\,\eta{\wedge}\bar\eta\\ -\beta{\wedge}\theta - \alpha{\wedge}\eta\\ -\imath\,\beta{\wedge}\bar\eta - 2\imath\,\bar\beta{\wedge}\eta - \imath\,\tau{\wedge}\theta\\ -\beta{\wedge}\bar\alpha - \imath\,\tau{\wedge}\eta\end{pmatrix}.$$

The above congruence for dR now implies that $\tau = a\,\theta + b\,\eta + \bar b\,\bar\eta$ for some real function a and complex function b on B_2. By adding $\imath a\,\eta$ to σ_0^2, I can arrange that $a = 0$, but I cannot absorb b. The last two structure equations now read

$$\begin{aligned} d\alpha &= -\sigma_0\wedge\theta - \imath\,\beta\wedge\bar\eta - 2\imath\,\bar\beta\wedge\eta - \imath\,(b\,\eta + \bar b\,\bar\eta)\wedge\theta\\ d\beta &= -\sigma_0^2\wedge\theta - \sigma_0\wedge\eta - \beta\wedge\bar\alpha + \imath\,\bar b\,\eta\wedge\bar\eta \end{aligned}$$

and it remains to determine how b varies on the fibers of $B_2 \to B_1$. To do this, compute the exterior derivative of the first of these equations and write it in the form

$$\begin{aligned} 0 = d(d\alpha) = &-\Big(d\sigma_0 - (\alpha+\bar\alpha)\wedge\sigma_0 - \imath\,\beta\wedge\bar\beta + \tfrac12\imath\big(\sigma_0^2\wedge\bar\eta - \overline{\sigma_0^2}\wedge\eta\big)\Big)\wedge\theta\\ &-\imath\,\Big(db - (2\alpha+\bar\alpha)\,b + \tfrac32\,\overline{\sigma_0^2}\Big)\wedge\eta\wedge\theta\\ &-\imath\,\big(d\bar b - (2\bar\alpha+\alpha)\,\bar b + \tfrac32\,\sigma_0^2\,\big)\wedge\bar\eta\wedge\theta \end{aligned}$$

The imaginary part of this equation implies that

$$db \equiv (2\alpha+\bar\alpha)\,b - \tfrac32\,\overline{\sigma_0^2} \mod \theta, \eta, \bar\eta,$$

which implies that, on each fiber of $B_2 \to B_1$, an equation of the form $db = \frac32\,\overline{ds^2}$ holds. In particular, the equation $b = 0$ defines a G_3-structure $B_3 \subset B_2$ on B_1 where G_3 is the 1-dimensional subgroup of G_2 defined by the equation $s^2 = 0$.

Now pull back all of the forms and functions involved to B_3. The structure equations take the form

$$d\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\ 0&0&0&0\\ \sigma_0&0&0&0\\ 0&\sigma_0&0&0\end{pmatrix}\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha){\wedge}\theta + \imath\,\eta{\wedge}\bar\eta\\ -\beta{\wedge}\theta - \alpha{\wedge}\eta\\ -\imath\,\beta{\wedge}\bar\eta - 2\imath\,\bar\beta{\wedge}\eta\\ -\beta{\wedge}\bar\alpha - \sigma_0^2{\wedge}\theta\end{pmatrix}.$$

where σ_0^2 is now basic, and, in fact, from the imaginary part of the equation above, must satisfy $\big(\overline{\sigma_0^2}{\wedge}\eta + \sigma_0^2{\wedge}\bar\eta\big){\wedge}\theta = 0$. This implies $\sigma_0^2{\wedge}\theta = (r\eta + s\bar\eta){\wedge}\theta$ where r and s are real and complex functions, respectively, on B_3. By adding $r\theta$ to σ_0 and calling the result σ, I can arrange that $r = 0$, and the structure equations become

$$d\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\ 0&0&0&0\\ \sigma&0&0&0\\ 0&\sigma&0&0\end{pmatrix}\begin{pmatrix}\theta\\ \eta\\ \alpha\\ \beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha){\wedge}\theta + \imath\,\eta{\wedge}\bar\eta\\ -\beta{\wedge}\theta - \alpha{\wedge}\eta\\ -\imath\,\beta{\wedge}\bar\eta - 2\imath\,\bar\beta{\wedge}\eta\\ -\beta{\wedge}\bar\alpha - s\,\bar\eta{\wedge}\theta\end{pmatrix}.$$

where, now σ is uniquely specified by these conditions. Thus, B_3 is endowed with a canonical $\{e\}$-structure and this constitutes the solution of the equivalence problem.

To complete the structure equations, however, a formula for $d\sigma$ is needed. The $d(d\alpha) = 0$ equation now yields

$$0 = \big(d\sigma - (\alpha + \bar\alpha)\wedge\sigma - \imath\,\beta\wedge\bar\beta\,\big)\wedge\theta$$

so that $d\sigma = (\alpha+\bar\alpha)\wedge\sigma + \imath\,\beta\wedge\bar\beta + \rho\wedge\theta$ where ρ is a real 1-form. Using this equation, the identity $d(d\beta) = 0$ expands to

$$0 = d(d\beta) = \rho\wedge\eta\wedge\theta - \big(ds - (3\bar\alpha+\alpha)s\big)\wedge\bar\eta\wedge\theta,$$

from which it follows that there are complex functions u, p, and q on B_3 so that

$$ds = (3\bar\alpha + \alpha)s + u\,\theta + p\,\eta + q\,\bar\eta$$

whence $\rho\wedge\theta = -(p\,\bar\eta + \bar p\,\eta)\wedge\theta$, so that

$$d\sigma = (\alpha+\bar\alpha)\wedge\sigma + \imath\,\beta\wedge\bar\beta - (p\,\bar\eta + \bar p\,\eta)\wedge\theta.$$

The final Bianchi identity will follow from $d\big(d\sigma\big) = 0$, and this expands to give the statement that there exist functions a, r, and v on B_3, with r being real valued, so that

$$dp = (3\bar\alpha + 2\alpha)p - \imath s\,\bar\beta + a\theta + r\eta + v\bar\eta.$$

3.2.3 *Conclusions.* Several conclusions can be drawn from these calculations.

First of all, since the group of symmetries of a non-degenerate CR-structure on a 3-manifold embeds into the group of symmetries of an $\{e\}$-structure on an 8-manifold, it follows that the group of symmetries of such a CR-structure is a Lie group of dimension at most 8.

Moreover, this maximum dimension can be reached only if the local symmetry group of the $\{e\}$-structure on B_3 acts with open orbits on B_3. However, examining the structure equations, this happens if and only if the functions s and p are locally constant. The structure equation for ds, however, shows that s cannot be locally constant unless it vanishes, which implies in turn that p vanishes as well. Then the equations

$$\begin{aligned}
d\theta &= -(\alpha+\bar\alpha)\wedge\theta + \imath\,\eta\wedge\bar\eta\\
d\eta &= -\beta\wedge\theta - \alpha\wedge\eta\\
d\alpha &= -\sigma\wedge\theta - \imath\,\beta\wedge\bar\eta - 2\imath\,\bar\beta\wedge\eta\\
d\beta &= -\sigma\wedge\eta + \bar\alpha\wedge\beta\\
d\sigma &= (\alpha+\bar\alpha)\wedge\sigma + \imath\,\beta\wedge\bar\beta
\end{aligned}$$

are the structure equations of a Lie group of dimension 8.

Naturally, the reader will want to know which one. The simplest way to identify the group is to notice that there are no $\alpha\wedge\bar\alpha$ terms on the right hand side of these equations but that α appears in the right hand side of all the equations except that of $d\alpha$. This implies that the vector fields X and Y dual to the real and imaginary parts of α form a maximal torus of dimension 2 in the Lie algebra of infinitesimal symmetries of the coframing. For any form ϕ in the coframing, define its X- and Y-weights by the formulae

$$\begin{aligned}
w_X(\phi)\,\phi &= X \lrcorner\, d\phi,\\
\imath\,w_Y(\phi)\,\phi &= Y \lrcorner\, d\phi.
\end{aligned}$$

Then plotting the pairs $\bigl(w_X(\phi), w_Y(\phi)\bigr)$ in the plane as ϕ ranges over the basis $(\theta, \eta, \bar\eta, \alpha, \bar\alpha, \beta, \bar\beta, \sigma)$ reveals the characteristic hexagon of the roots of A_2. Moreover, because the roots are 'half-real', the actual real form of A_2 represented must be $\mathfrak{su}(2,1)$. Thus, the group must be $\mathrm{SU}(2,1)$. In fact, the structure equations can be written in the form $d\gamma = -\gamma{\wedge}\gamma$ where

$$\gamma = \begin{pmatrix} -\frac{1}{3}(2\alpha+\bar\alpha) & -\imath\,\bar\beta & -\imath\,\sigma \\ \eta & \frac{1}{3}(\alpha-\bar\alpha) & \imath\,\beta \\ -\imath\,\theta & \bar\eta & \frac{1}{3}(\alpha+2\bar\alpha) \end{pmatrix}$$

Note that γ takes values in $\mathfrak{su}(2,1)$, where the model of $\mathrm{SU}(2,1)$ being used is the subgroup of $SL(3,\mathbb{C})$ that fixes the Hermitian form H in three variables $H(Z) = Z_3\,\overline{Z_1} - Z_2\,\overline{Z_2} + Z_1\,\overline{Z_3}$.

In particular, if M^3 is simply connected, there is a smooth map $F : B_3 \to \mathrm{SU}(2,1)$ that satisfies $F^*(g^{-1}\,dg) = \gamma$. As the structure equations reveal, F maps each fiber of $B_3 \to M$ to a left coset of the parabolic subgroup $P \subset \mathrm{SU}(2,1)$ consisting of the upper triangular matrices in $\mathrm{SU}(2,1)$, i.e., the subgroup that fixes the H-null line $L_0 \subset \mathbb{C}^3$ defined by $Z_2 = Z_3 = 0$. Thus, $\mathrm{SU}(2,1)/P$ is naturally identified with the hypersurface $N^3 \subset \mathbb{CP}^2$ of H-null lines in $\mathbb{CP}^2$. Thus, F covers a map $f : M^3 \to N^3$ that is a local equivalence of CR-structures. The conclusion is that every CR-structure with 8-dimensional infinitesimal symmetry algebra on a simply connected 3-manifold has a 'developing map' to N^3 that is unique up to composition with a CR-automorphism of this 'flat' structure.[4]

Second, in the general case, s is the coefficient of a tensor field that is well-defined on M. The simplest such expression involving s is perhaps $Q = s\bar s\,\theta^4$, which is a well-defined section of $S^4\bigl(D^\perp\bigr)$ on M. This follows since Q is manifestly semibasic and and a computation using the structure equations reveals that, for any vertical vector field Y for the projection $B_3 \to M$, the Lie derivative of Q with respect to Y vanishes. One can also interpret the expression $S = s\,\bar\eta{\otimes}\bar\eta{\otimes}\theta$ as a well-defined section of the bundle $S^{0,2}(D){\otimes}D^\perp$ over M, i.e., the bundle of complex anti-linear quadratic forms on D with values in $D^\perp$. Other combinations of the functions on B_3 make well-defined tensors on M as well, but have to be treated with more care. For example, the expression $E = s\,\bar\eta^2{\circ}\theta + 2\imath p\,\bar\eta{\circ}\theta^2$ mod θ^3 yields a well-defined section of the quotient bundle $S^3(T^*M)/(D^\perp)^3$. The verification of these statements will be left to the reader.

Third, in the case where $Q = s\bar s\,\theta^4$ is non-vanishing on M, there is a canonical reduction of B_3 to a $\mathbb{Z}_2$-structure $B_4 \to M$ defined by the equations $s = -1$, $p = 0$, $u + \bar u = 0$. This follows from the formulae for ds and dp together with the formula

$$du \equiv (4\bar\alpha + 2\alpha)u + p\,\beta + 4s\sigma \mod \theta, \eta, \bar\eta$$

(which is derived from the identity $d(ds) = 0$). Pulling all the given quantities back to B_4, writing $u = 2\imath m$ where m is real and replacing q by $8\bar q$ for notational convenience, this results in equations

$$\alpha = \imath m\,\theta - 3q\,\eta + \bar q\,\bar\eta$$
$$\beta = \imath a\,\theta + \imath v\,\eta + \imath r\,\bar\eta$$

[4]Explicitly computing this developing map requires solving a Lie equation of the form $dg = g\gamma$ where γ is a known 1-form with values in $\mathfrak{su}(2,1)$.

resulting in structure equations of the form

$$
\begin{aligned}
d\theta &= 2(q\eta + \bar{q}\bar{\eta}) \wedge \theta + \imath\,\eta \wedge \bar{\eta} \\
d\eta &= t\,\eta \wedge \theta + \bar{q}\,\eta \wedge \bar{\eta} - \imath r\,\bar{\eta} \wedge \theta
\end{aligned}
$$

for some function t constructed out of the other invariants. Under the $\mathbb{Z}_2$-action on the double cover $B_4 \to M$, the form θ is even while η is odd. Thus, the coframining (θ, η) is well-defined on M up to a replacement of the form $(\theta, \eta) \mapsto (\theta, -\eta)$. It also follows that t and r are even while q is odd.

In particular, it follows from this that the group of symmetries of a non-degenerate CR-structure for which $Q \neq 0$ is a Lie group of dimension at most 3 and that this upper bound is reached only for homogeneous structures, in which case, the functions q, r, and t must be constants. Indeed, if one assumes that these functions are constants, then computing the exterior derivatives of the above equations yields that $t + \bar{t} = 0$, so that $t = \imath b$, for some real constant b, and the equation $rq + b\bar{q} = 0$. Conversely, any solution $(r, b, q) \in \mathbb{R}^2 \times \mathbb{C}$ of $rq + b\bar{q} = 0$ defines a homogeneous CR-structure. Cartan used this fact in his original paper [**Ca??**] to classify the homogeneous CR-hypersurfaces in $\mathbb{C}^2$.

3.3. **Monge-Ampére systems in two independent variables.** A long example explaining the geometry of Monge-Ampére systems on 5-manifolds and explaining the three types.

3.4. **Monge-Ampére systems in three independent variables.** A long example explaining the geometry of Monge-Ampére systems on 7-manifolds and explaining the algebra of constant types together with a first pass at the invariants.

3.5. **Almost complex 4-manifolds.**

3.6. **Pfaffian systems.**